Thin DLPP in KPZ universality

Yuxuan Zong

USTC

January 2, 2023

Overview

Overview

1. Introduction and history of DLPP
2. Limiting distribution of thin DLPP
3. Some open problems in DLPP

The KPZ universality is mainly concerned about the fluctuations of these models:

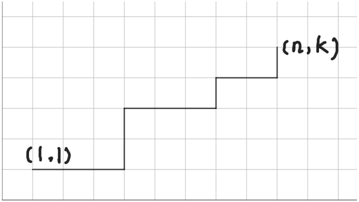
- (1+1)d random growth (e.g. corner growth)
- 1d interacting particle system (e.g. TASEP)
- 2d directed polymers (e.g. directed last passage percolation)

KPZ universality conjecture: For height function $h(x,t)$, as $T \to +\infty$,

$$\frac{h(x_0 T + C_1 x T^{2/3}, tT) - a(T)}{C_2 T^{1/3}} \Rightarrow \mathcal{H}(x,t)$$

with some model-independent constants C_1, C_2 and $a(T)$, and a universal 2d random field $\mathcal{H}(x,t)$, which is now called the **KPZ fixed point**.

Scaling: fluctuation : space : time = 1 : 2 : 3



DLPP

- i.i.d. random variables w_{ij} at each site $(i, j) \in \mathbb{Z}^2$. Let $\mu = \mathbb{E}[w_{ij}], \mathrm{Var}(w_{ij}) = \sigma^2$.
- Admissible paths: $\Pi(n, k)$= the set of "up/right" paths from $(1, 1)$ to (n, k).
- Last passage time: $T(n, k) = \max_{\pi \in \Pi(n,k)} \left\{ \sum_{(i,j) \in \pi} w_{ij} \right\}$.
- Related models: Corner growth, TASEP, $\cdots$

Expectation of DLPP

- $k = o(n)$:

Theorem (Seppaäläinen, 1997)

$$\lim_{n\to+\infty} \frac{\mathbb{E}[T(n,[n^a])] - n\mu}{n^{\frac{1+a}{2}}} = 2\sigma, \quad a \in (0,1).$$

- $k = O(n)$, We have consequences of some solvable models:

$$\lim_{n\to+\infty} \frac{\mathbb{E}[T([xn],n)]}{n} = a(x).$$

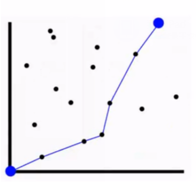
Theorem (Rost, 1981)

For $w_{ij} \sim Exp(1), a(x) = (1+\sqrt{x})^2$.

Theorem (Johansson, 2000)

For $w_{ij} \sim Geo(q), 0 < q < 1, a(x) = \frac{q(1+x)+2\sqrt{qx}}{1-q}$.

Poisson DLPP:



Poisson DLPP

- A 2d Poisson process in $\mathbb{R}_+^2$;
- Admissible paths: $\Pi(t,s)=$ the set of upper right path p that connects Poisson points linearly, $(t,s) \in \mathbb{R}_+^2$;
- $E(p)$: the number of the Poisson points on p.
- Last passage time: $T(t,s) = \sup_{p \in \Pi(t,s)} E(p)$.

Theorem (Baik-Deift-Johansson, 1999)

$$\frac{T(n,n) - 2n}{n^{1/3}} \xrightarrow{D} F_{TW}, \quad n \to +\infty.$$

Exponential, Geometric DLPP:

Theorem (Johansson, 2000)

For $w_{ij} \sim Exp(1)/Geo(q)$, we have that

$$\frac{T([xn], n) - a(x)n}{b(x)n^{1/3}} \xrightarrow{D} F_{TW}, \quad n \to +\infty.$$

- $Exp(1) : b(x) = \dfrac{(1+\sqrt{x})^{4/3}}{x^{1/6}}$;
- $Geo(q) : b(x) = \dfrac{q^{1/6}}{(1-q)x^{1/6}}(\sqrt{x}+\sqrt{q})^{2/3}(1+\sqrt{qx})^{2/3}$.

Question 1: Is it similar for general i.i.d w_{ij} ?

For **thin DLPP**, **Yes** !

Question 2: How thin it is ? All k=o(n)?

At least $k = n^a$, $a < 3/7$.

Overview

Theorem (Baik-Suidan & Bodineau-Martin , 2005)

Suppose $\mu = 0, \sigma = 1$, w_{ij} has finite p moments, $p > 2$, we have that

$$\frac{T(n,k) - 2\sqrt{nk}}{n^{1/2}k^{-1/6}} \xrightarrow{D} F_{TW}, \quad n, k \to +\infty$$

with $k = [n^a]$, $0 < a < \frac{3(p-2)}{7p}$

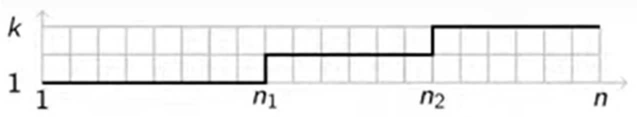
Remark

If w_{ij} has all moments, then $0 < a < 3/7$ is enough.

Motivation

- Fix k , $n \to +\infty$: each level $\approx$ Brownian motion. Note that

$$T(n,k) = \max_{1=n_0 \le n_1 \le \cdots \le n_k = n} \sum_{i=1}^{n} (S^{(i)}_{n_i+1} - S^{(i)}_{n_i}), \ \ S_j(i) = \sum_{m=0}^{i-1} w_{mj}.$$



Let

$$D(t,k) = \sup_{0=t_0 \le t_1 \le \cdots \le t_k = t} \sum_{i=1}^{k} (B^{(i)}_{t_i} - B^{(i)}_{t_{i-1}}),$$

where $\left\{ B^{(i)}_t, t \ge 0, i \ge 1 \right\}$ be a sequence of independent standard Brownian motions. By Donsker's theorem, we can imagine that

$$\frac{T(n,k)}{\sqrt{n}} \xrightarrow{D} D(1,k), \quad n \to +\infty.$$

This was proved by Glynn and Whitt in 1991.

Theorem (Tracy-Widom, 1994)

Let λ_{max} be the largest eigenvalue of $k \times k$ GUE random matrix, then

$$k^{1/6}(\lambda_{max} - 2\sqrt{k}) \xrightarrow{D} F_{TW}, \quad k \to +\infty.$$

- (Baryshnikov & Gravner-Tracy-Widom, 2001) $D(1,k)$ has the same distribution as the largest eigenvalue of a $k \times k$ GUE random matrix, so

$$k^{1/6}(D(1,k) - 2\sqrt{k}) \xrightarrow{D} F_{TW}, \quad k \to +\infty.$$

Therefore,

$$\lim_{k\to+\infty} \lim_{n\to+\infty} \mathbb{P}\left(k^{1/6}\left\{\frac{T(n,k)}{\sqrt{n}} - 2\sqrt{k}\right\} \leq s\right) = F_{TW}(s).$$

- Question: $\lim_{k\to+\infty} \lim_{n\to+\infty} \stackrel{?}{=} \lim_{n,k\to+\infty}$
- Approach: couple Brownian motions.
  Baik-Suidan: Skorohod Embedding;
  Bodineau-Martin: KMT (Komlós-Major-Tusnády) approximation.

Scaling:

$$k^{1/6}(D(1,k) - 2\sqrt{k}) \xrightarrow{D} F_{TW}$$

$$\downarrow$$

$$\frac{D(n,k) - 2\sqrt{nk}}{n^{1/2}k^{-1/6}} \xrightarrow{D} F_{TW}$$

So we need to prove that

$$\frac{|T(n,k) - D(n,k)|}{n^{1/2}k^{-1/6}} \xrightarrow{P} 0, \quad n \to +\infty$$

with $k = [n^a]$, $0 < a < \frac{3(p-2)}{7p}$.

Proposition (Skorohod Embedding Theorem)

Let $\{B_t\}_{t\in\mathbb{R}^+}$ be a 1d standard Brownian motion and X be a real valued random variable satisfying $\mathbb{E}[X]=0, \mathbb{E}[X^2]=1$, and $\mathbb{E}[X^4]<\infty$. Then, there exists a Brownian stopping time τ, such that B_τ is distributed as X, $\mathbb{E}[\tau]=\mathbb{E}[X^2]=1$, and $\mathbb{E}[\tau^2]\leq 4\mathbb{E}[X^4]$.

Proposition (KMT approximation)

Suppose $w_i, i=1,2,\cdots$ i.i.d, with $\mathbb{E}|w_i|^p<\infty$ for some $p>2$, and with $\mathbb{E}[w_i]=0, \mathrm{Var}(w_i)=1$. Let $S_m=\sum_{i=0}^{m-1} w_i, m\geq 1$. Then there is a constant C such that for all $n>0$, there is a coupling of the distribution of $(w_1,\cdots,w_n)$ and a standard Brownian motion B_t, $0\leq t\leq n+1$ such that, for all $x\in[n^{1/p},n^{1/2}]$,

$$\mathbb{P}\left(\max_{m=1,2,\cdots n+1}|B_m-S_m|>x\right)\leq Cnx^{-p}.$$

Bodineau-Martin‘s proof

Define

$$\mathcal{U}(t,k)=\left\{(u_0,u_1,\cdots u_k)\in\mathbb{R}^{k+1}:0=u_0\le u_1\le\cdots u_k=t\right\},$$

we have that

$$\begin{aligned}
|T(n,[n^a])-D(n,[n^a])| &= \left|\sup_{\boldsymbol{u}\in\mathcal{U}(n,[n^a])}\sum_{r=1}^{[n^a]}(S^{(r)}_{[u_r]+1}-S^{(r)}_{[u_{r-1}]})-\sup_{\boldsymbol{u}\in\mathcal{U}(n,[n^a])}\sum_{r=1}^{[n^a]}(B^r_{u_r+1}-B^r_{u_{r-1}}))\right| \\
&\le \sup_{\boldsymbol{u}\in\mathcal{U}(n,[n^a])}\sum_{r=1}^{[n^a]}(|S^{(r)}_{[u_r]+1}-B^{(r)}_{[u_r]+1}|+|S^{(r)}_{[u_{r-1}]}-B^{(r)}_{[u_{r-1}]}| \\
&+|B^{(r)}_{[u_r]+1}-B^{(r)}_{u_r}|+|B^{(r)}_{[u_{r-1}]}-B^{(r)}_{u_{r-1}}|) \\
&\le 2\sum_{r=1}^{[n^a]}\left(\max_{i=1,2,\cdots n+1}|S^{(r)}_i-B^{(r)}_i|+\sup_{\substack{0\le s,t\le n+1\\|s-t|<2}}|B^{(r)}_s-B^{(r)}_t|\right) \\
&:= 2\sum_{r=1}^{[n^a]}(V^{(r)}_n+W^{(r)}_n)
\end{aligned}$$

Let event $A_1 := \left\{ \max_{1\le r\le [n^a]} V_n^{(r)} > n^{1/2} \right\}$, event $A_2 := \left\{ \max_{1\le r\le [n^a]} W_n^{(r)} > n^{1/p} \right\}$

- For each $r = 1, \cdots, [n^a]$ we will couple $(w_0^{(r)}, w_1^{(r)}, \cdots, w_n^{(r)})$ and $\left\{ B_t^{(r)}, 0 \le t \le n+1 \right\}$. By KMT approximation,

$$\mathbb{P}(A_1) \le n^a \cdot Cn(n^{1/2})^{-p} = Cn^{a+1-p/2} \xrightarrow{n\to+\infty} 0,$$

since

$$a < \frac{6}{7}\left(\frac{1}{2} - \frac{1}{p}\right) < p\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{p}{2} - 1.$$

- By reflection principle and standard estimates on the normal distribution,

$$\mathbb{P}(A_2) \le n^a \mathbb{P}\left(\sup_{\substack{0\le s,t\le n+1 \\ |s-t|<2}} |S_s^{(1)} - B_t^{(1)}| \right) \le n^a \sum_{i=0}^{n-2} \mathbb{P}(\sup_{i\le t\le i+3} B_t - \inf_{t\le i\le t+3} B_t > n^{1/p})$$

$$\le n^{a+1}\mathbb{P}\left(\sup_{0\le t\le 3} |B_t| > \frac{n^{1/p}}{2} \right) = 2n^{a+1}\mathbb{P}\left(B_3 > \frac{n^{1/p}}{2} \right)$$

$$\le C_1 n^{a+1} \exp\left(-C_2 n^{-2/p} \right) \xrightarrow{n\to+\infty} 0.$$

- Finally,

$$\begin{aligned}\mathbb{E}[|T(n,[n^a])-D(n,[n^a])|;A_1^c\cap A_2^c] &\le 2n^a\mathbb{E}[V_n^{(1)}+W_n^{(1)};A_1^c\cap A_2^c]\\ &\le 2n^a[2n^{1/p}+\mathbb{E}[V_n^{(1)}-n^{1/p};n^{1/p}\le V_n^{(1)}\le n^{1/2}]]\\ &\le 2n^a\left(2n^{1/p}+\int_{n^{1/p}}^{n^{1/2}}\mathbb{P}(V_n^{(1)}>x)\mathrm{d}x\right)\\ &\le 2n^a\left(2n^{1/p}+\int_{n^{1/p}}^{n^{1/2}}Cnx^{-p}\mathrm{d}x\right)\\ &\le C_3n^{a+1/p}.\end{aligned}$$

Above all, for any $\varepsilon>0$,

$$\mathbb{P}(|T(n,[n^a])-D(n,[n^a])|>n^{a+1/p+\varepsilon})\xrightarrow{n\to+\infty}0.$$

Since $a<\frac{6}{7}\left(\frac{1}{2}-\frac{1}{p}\right)$, for ε sufficiently small, $a+\frac{1}{p}+\varepsilon<\frac{1}{2}-\frac{a}{6}$. Therefore,

$$\frac{|T(n,[n^a])-D(n,[n^a])|}{n^{1/2-a/6}}\xrightarrow{P}0,\quad n\to+\infty.$$

Overview

- Is the above limiting distribution right for larger values of a? Even any $k, n \to +\infty$?

  A possible try: Four moments theorem.

  cf:
  Tao and Vu,Random matrices: The Four Moment Theorem for Wigner ensembles, 2014.
  Adhikari and Chatterjee, An invariance principle for the 1D KPZ equation, 2022.

- For general iid weight, including k=O(n) and k=o(n), limit shape? fluctuation?

  Better tail bounds: Shirshendu Ganguly, Optimal tail exponents in general last passage percolation via bootstrapping & geodesic geometry, Milind Hegde, 2020.

References

- J. Baik and T. M. Suidan, A GUE central limit theorem and universality of directed first and last passage site percolation. Int. Math. Res. Not. , no. 6, 325–337, 2005.
- T. Bodineau and J. Martin, A universality property for last-passage percolation paths close to the axis. Electron. Comm. Probab. 10, 105–112, 2005.
- J. Baik, KPZ limit theorems, 2022, arXiv:2206.14086.
- Komlós, J., Major, P. and Tusnády, G., An approximation of partial sums of independent RV's, and the sample DF. II. Z., Wahrscheinlichkeitstheorie und Verw. Gebiete 34, 33–58, 1976.
- Y. Baryshnikov, GUEs and queues. Probab. Theory Related Fields 119, no. 2, 256–274, 2001.
- J. Gravner, C. Tracy, and H. Widom, Limit theorems for height fluctuations in a class of discrete space and time growth models, J. Statist. Phys. 102 , no. 5-6, 1085–1132, 2001.
- P. Glynn and W. Whitt, Departures from many queues in series, Ann. Appl. Probab. 1 , no. 4, 546–572, 1991.
- C. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, Comm. Math. Phys.159, no. 1, 151–174, 1994.

- J. Baik, P. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12, no. 4, 1119–1178, 1999.
- K. Johansson, Shape fluctuations and random matrices, Comm. Math. Phys. 209, no. 2, 437–476, 2000.
- H. Rost, Nonequilibrium behaviour of a many particle process: density profile and local equilibria, Z. Wahrsch. Verw. Gebiete 58, no. 1, 41–53, 1981.
- T. Seppaäläinen , A scaling limit for queues in series, Ann. Appl. Probab. 7, no. 4, 855–872, 1997.
- T.Tao and V.Vu, The Four Moment Theorem for Wigner ensembles, Math. Sci. Res. Inst. Publ., 65 , Cambridge Univ. Press, 509–528, 2014.
- Arka Adhikari and Sourav Chatterjee, An invariance principle for the 1D KPZ equation, 2022, arxiv:2208.02492.
- M. Hegde and S. Ganguly, Optimal tail exponents in general last passage percolation via bootstrapping & geodesic geometry, 2020, arXiv:2007.03594

Thank you!